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The Drinfeld double $gl(n) \oplus t_n$

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Abstract
We present a self-dual Drinfeld double structure underlying the A_n series of simple Lie algebras. Such double is constructed through a central extension t_n of $gl(n)$, and is obtained by pairing two disjoint solvable subalgebras coming from positive and negative roots. The Cartan–Weyl basis of $gl(n)$ is shown to be completely determined by the compatibility conditions in the double. A natural Lie bialgebra structure on $gl(n)$ is obtained that offers a new perspective for the construction of its quantum deformations.

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1. Introduction

The concept of Drinfeld double is just a reformulation of that of Lie bialgebra in terms of a (double) dimensional Lie algebra endowed with a suitable pairing. Moreover, a well-known result by Drinfeld states that Lie bialgebras are in one-to-one correspondence with Poisson–Lie groups [1, 2].

These facts turn out to be relevant from a physical viewpoint, since from a given Drinfeld double a pair of σ-models related by Poisson–Lie T-duality can be constructed (see [3–6] and references therein). Thus, any new example of the Drinfeld double would be interesting, but its explicit construction and classification is a difficult problem. In fact, only 4- and 6-dimensional Drinfeld doubles have been fully described so far [7–9].

On the other hand, the Hopf algebra quantization of a Drinfeld double is the so-called quantum double, a basic object in quantum group theory (see, for instance, [2, 10–14] for a detailed exposition and references therein). In particular, quantum doubles play an essential role in the explicit construction of quantum R-matrices, which are cornerstones of the theory of quantum integrable models (see [15–22]). Quantum doubles have also been considered as symmetries in quantum field theory [23, 24].

The aim of this paper is to provide explicitly a family of $n(n+1)$-dimensional Drinfeld doubles $\bar{g} = gl(n) \oplus t_n$ (where t_n is an Abelian algebra) which are directly related to the A_n Cartan series. From a general viewpoint, this result will allow us to reinterpret simple Lie algebras as Drinfeld doubles.

The Drinfeld double structure will be obtained by enlarging the Cartan subalgebra of the algebra $gl(n)$ in such a way that two disjoint solvable algebras, isomorphic to Borel subalgebras, can be properly paired as follows:

- Firstly, the n-dimensional Cartan subalgebra h_n is enlarged by introducing its direct sum with an additional Abelian algebra t_n generated by I_i, $i = 1, \ldots, n$. A new basis in the $2n$-dimensional Abelian algebra $h_n \oplus t_n$ is thus defined as

$$X_i := \frac{1}{\sqrt{2}}(H_i + \mathrm{i}I_i), \qquad x^i := \frac{1}{\sqrt{2}}(H_i - \mathrm{i}I_i), \tag{1.1}$$

where i is the imaginary unit.
- Afterwards, we consider two disjoint and isomorphic solvable Lie algebras s_+ and s_-, which contain the X_i's and x^i's generators and the positive and negative roots of $gl(n)$, respectively.
- Finally, we check that these two subalgebras, together with their crossed commutation rules, define a Drinfeld double structure on $\bar{g} = gl(n) \oplus t_n$.

As a result of this construction, the Drinfeld double seems to be a structural ingredient of simple Lie algebras. In order to define such self-dual Drinfeld double we find that the usual description of simple Lie algebras in terms of the Chevalley–Cartan basis (and, obviously, Serre relations) is not suitable, and we are forced to use a Cartan–Weyl one. Moreover, the freedom in the choice of basis in the Cartan subalgebra and the indetermination given by the nonfixed length of the root vectors in the Cartan approach (see [25]) is removed by imposing the existence of a Drinfeld double structure that up to a unique scale factor requires an orthonormal basis in the Cartan subalgebra and fixes all the commutation rules. The Cartan–Weyl basis is, hence, completely determined by its underlying Drinfeld double structure.

We also recall that, from a quantum point of view, Drinfeld–Jimbo deformations of semi-simple Lie algebras [26, 27] are closely related to quantum doubles. For a complete discussion of the problem we refer to [13], where it is shown that for every finite-dimensional complex simple Lie algebra g, its standard quantization $U_h(g)$ is 'almost' a quantum double. However, since the positive and negative quantum Borel subalgebras have in common the Cartan subalgebra, the pairing between them cannot exist and a Drinfeld double structure is not available. We stress that the results presented here also provide an explicit way to circumvent this problem and to construct a family of quantum deformations of simple algebras as true quantum doubles.

The paper is organized as follows. In section 2 we recall the basic definitions. Section 3 is devoted to the $gl(2)$ case, which is fully discussed. The generalization to $gl(n)$ is presented in section 4 and some comments and remarks are included in the last section.

2. Manin triples and Drinfeld doubles

We recall that a Lie algebra $\bar{g}$ is called a Drinfeld double if it can be endowed with a Manin triple structure [2, 13, 14]. A Manin triple is a set of three Lie algebras $(s_+, s_-, \bar{g})$ such that s_+ and s_- are disjoint subalgebras of $\bar{g}$ having the same dimension, $\bar{g} = s_+ + s_-$ as vector spaces and the crossed commutation rules between s_+ and s_- are defined in terms of the structure tensors f and c of s_+ and s_-. Explicitly, we consider

$$[Z_p, Z_q] = f^r_{p,q} Z_r, \qquad Z_p, Z_q, Z_r \in s_+, \tag{2.2}$$

$$[z^p, z^q] = c^{p,q}_r z^r, \qquad z^p, z^q, z^r \in s_-. \tag{2.3}$$

Now let us define a pairing (i.e., a non-degenerate symmetric bilinear form on the vector space $s_+ \oplus s_-$ for which $s_\pm$ are isotropic)

$$\langle Z_p, Z_q \rangle = 0, \qquad \langle Z_p, z^q \rangle = \delta^q_p, \qquad \langle z^p, z^q \rangle = 0. \tag{2.4}$$

If the compatibility relations (crossed Jacobi identities)

$$c^{p,q}_r f^r_{s,t} = c^{p,r}_s f^q_{r,t} + c^{r,q}_s f^p_{r,t} + c^{p,r}_t f^q_{s,r} + c^{r,q}_t f^p_{s,r} \tag{2.5}$$

are fulfilled, we can construct a new Lie algebra $\bar{g}$ such that, as a vector space, $\bar{g} = s_+ \oplus s_-$ and such that the pairing is invariant under the adjoint representation of $\bar{g}$ (i.e., $\langle [a, b], c \rangle = -\langle a, [b, c] \rangle], \forall a, b, c \in \bar{g}$). The latter condition leads to the following crossed commutation rules between the elements of s_+ and s_-:

$$[z^p, Z_q] = f^p_{q,r} z^r - c^{p,r}_q Z_r. \tag{2.6}$$

The Lie algebra $\bar{g}$ is then called a Drinfeld double and $(s_+, s_-, \bar{g})$ is called a Manin triple. Then $\bar{g}$ can be endowed with a (quasitriangular) Lie bialgebra structure $(\bar{g}, \delta_D)$

$$\delta_D(Z_p) = -\eta(Z_p) = -c^{q,r}_p Z_q \otimes Z_r, \qquad \delta_D(z^p) = \delta(z^p) = f^p_{q,r} z^q \otimes z^r. \tag{2.7}$$

This ‘double Lie bialgebra’ has as Lie sub-bialgebras $(s_+, -\eta)$ and its dual (s_-, δ). Obviously, several Manin triple structures for a given $\bar{g}$ can be constructed (see, for instance, [9, 28, 29]).

We remark that (2.7) can be derived either from the classical r-matrix

$$r = \sum_p z^p \otimes Z_p,$$

or from its skew-symmetric counterpart

$$\tilde{r} = \frac{1}{2} \sum_p z^p \wedge Z_p. \tag{2.8}$$

If, by following [29], in an appropriate basis we have that $c = -f$ we shall say that $\bar{g}$ is a self-dual Drinfeld double. It seems clear that the positive and negative Borel subalgebras $b_\pm$ of any classical Lie algebra g have this property, but they do have the Cartan subalgebra in common and, therefore, $b_\pm$ cannot be identified as $s_\pm$. As a consequence, although the Weyl symmetry is present, a classical Lie algebra g is not a Drinfeld double [2, 13].

3. The Drinfeld double $gl(2) \oplus t_2$

In order to illustrate our construction, let us start with the elementary example of the Drinfeld double structure of the algebra $gl(2) \oplus t_2$.

We start by considering the three-dimensional solvable algebras $s^+ = \{Z_1, Z_2, Z_3\}$ and $s^- = \{z^1, z^2, z^3\}$ with commutation rules

$$[Z_1, Z_2] = 0, \qquad [Z_1, Z_3] = \tfrac{1}{\sqrt{2}} Z_3, \qquad [Z_2, Z_3] = -\tfrac{1}{\sqrt{2}} Z_3, \tag{3.9}$$

$$[z^1, z^2] = 0, \qquad [z^1, z^3] = -\tfrac{1}{\sqrt{2}} z^3, \qquad [z^2, z^3] = \tfrac{1}{\sqrt{2}} z^3. \tag{3.10}$$

The structure tensors for s_+, $f^p_{q,r}$ (2.2) and s_-, $c^{p,r}_q$ (2.3) are

$$f^3_{1,3} = -f^3_{3,1} = \tfrac{1}{\sqrt{2}}, \qquad f^3_{2,3} = -f^3_{3,2} = -\tfrac{1}{\sqrt{2}}, \qquad c^{p,q}_r = -f^r_{p,q}.$$

Now, let us consider the triple $(s_+, s_-, \bar{g} = s_+ + s_-)$ and a bilinear form on $\bar{g}$ defined through (2.4). Compatibility conditions (2.5) are easily checked and the crossed commutation rules between s_+ and s_- are given by (2.6):

$$
\begin{aligned}
&[z^1, Z_3] = -[z^2, Z_3] = \tfrac{1}{\sqrt{2}} Z_3,\\
&[z^3, Z_1] = -[z^3, Z_2] = \tfrac{1}{\sqrt{2}} z^3,\\
&[z^3, Z_3] = -\tfrac{1}{\sqrt{2}}(z^1 + Z_1) + \tfrac{1}{\sqrt{2}}(z^2 + Z_2).
\end{aligned}
\tag{3.11}
$$

Since s_+ and s_- are isomorphic, we are dealing with a self-dual Manin triple [29]. In other words, (s_+, η) is a Lie bialgebra with co-commutator

$$\eta(Z_p) = c_p^{q,r} Z_q \otimes Z_r.$$

Explicitly,

$$\eta(Z_1) = \eta(Z_2) = 0, \qquad \eta(Z_3) = \tfrac{1}{\sqrt{2}} Z_3 \wedge (Z_1 - Z_2).$$

Correspondingly, (s_-, δ) is the dual Lie bialgebra with co-commutator

$$\delta(z^p) = f_{q,r}^{p} z^q \otimes z^r,$$

which reads

$$\delta(z^1) = \delta(z^2) = 0, \qquad \delta(z^3) = -\tfrac{1}{\sqrt{2}} z^3 \wedge (z^1 - z^2).$$

Now, by considering the change of basis

$$
\begin{aligned}
&H_1 = \frac{1}{\sqrt{2}}(Z_1 + z^1), && I_1 = \frac{1}{\mathrm{i}\sqrt{2}}(Z_1 - z^1),\\
&H_2 = \frac{1}{\sqrt{2}}(Z_2 + z^2), && I_2 = \frac{1}{\mathrm{i}\sqrt{2}}(Z_2 - z^2),\\
&F_{12} = Z_3, && F_{21} = z^3,
\end{aligned}
\tag{3.12}
$$

and rewriting relations (3.9)–(3.11), we obtain

$$
\begin{aligned}
&[I_i, \cdot] = 0, && [H_1, H_2] = 0, && [H_1, F_{12}] = F_{12}, && [H_1, F_{21}] = -F_{21},\\
&[H_2, F_{12}] = -F_{12}, && [H_2, F_{21}] = F_{21}, && [F_{12}, F_{21}] = H_1 - H_2,
\end{aligned}
\tag{3.13}
$$

which are just the commutation rules for the Lie algebra $\bar{g} = gl(2) \oplus t_2$ in the usual basis $\{H_1, H_2, F_{12}, F_{21}\} \oplus \{I_1, I_2\}$.

Therefore, we have proven that the two solvable algebras s_+ and s_- together with the pairing (2.4) endow $\bar{g} = gl(2) \oplus t_2$ with a Drinfeld double structure. Note that s_+ and s_- have been chosen to be isomorphic to the upper and lower triangular 2×2 matrices of $gl(2)$, respectively.

Explicitly, the associated Lie bialgebra (2.7) is

$$
\begin{aligned}
&\delta_D(I_i) = 0,\\
&\delta_D(H_i) = 0,\\
&\delta_D(F_{12}) = -\frac{1}{2} F_{12} \wedge (H_1 - H_2) - \frac{\mathrm{i}}{2} F_{12} \wedge (I_1 - I_2),\\
&\delta_D(F_{21}) = -\frac{1}{2} F_{21} \wedge (H_1 - H_2) + \frac{\mathrm{i}}{2} F_{21} \wedge (I_1 - I_2).
\end{aligned}
$$

This co-commutator δ_D can be derived from the classical r-matrix (2.8), i.e.

$$\tilde{r} = \frac{1}{2} F_{21} \wedge F_{12} + \frac{\mathrm{i}}{4}(H_1 \wedge I_1 + H_2 \wedge I_2) = \tilde{r}_s + \tilde{r}_t,$$

where $\tilde{r}_s$ generates the standard deformation of $gl(2)$ and $\tilde{r}_t$ denotes a twist that becomes trivial in the representation of t_2 where $I_1 - I_2 = 0$.

It is worth noting that from (3.12) the pairing (2.4) is simply the Killing form relations for $gl(2)$ in the 'oscillator representation' convention [30]

$$\langle H_i, H_j \rangle = \delta_{ij}, \qquad \langle H_i, F_{jk} \rangle = 0, \qquad \langle F_{ij}, F_{kl} \rangle = \delta_{jk}\delta_{il} \tag{3.14}$$

supplemented by a suitable definition of the pairing for the additional central generators

$$\langle I_i, I_j \rangle = \delta_{ij}, \qquad \langle I_i, H_j \rangle = 0, \qquad \langle I_i, F_{jk} \rangle = 0. \tag{3.15}$$

4. The $gl(n) \oplus t_n$ case

For the general case of $gl(n)$ the procedure is similar to the one followed in the preceding case of $gl(2)$. We consider two $n(n+1)/2$-dimensional solvable Lie algebras s_+ and s_-, isomorphic to the subalgebras defined by upper and lower triangular $n \times n$ matrices of $gl(n)$, with generators

$$s^+ : \quad \{X_i, Y_{ij}\}, \qquad i, j = 1, \ldots, n, \quad i < j,$$
$$s^- : \quad \{x^i, y^{ij}\}, \qquad i, j = 1, \ldots, n, \quad i < j,$$

and commutation rules given by

$$[X_i, X_j] = 0, \qquad [X_i, Y_{jk}] = \tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})Y_{jk}, \qquad [Y_{ij}, Y_{kl}] = (\delta_{jk}Y_{il} - \delta_{il}Y_{kj}),$$
$$[x^i, x^j] = 0, \qquad [x^i, y^{jk}] = -\tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})y^{jk}, \qquad [y^{ij}, y^{kl}] = -(\delta_{jk}y^{il} - \delta_{il}y^{kj}).$$

Following (2.2) and (2.3), the corresponding structure tensors f and c read

$$f_{i,jk}^{lm} = -f_{jk,i}^{lm} = -c_{lm}^{i,jk} = c_{lm}^{jk,i} = \tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})\delta_{jl}\delta_{km},$$
$$f_{ij,kl}^{mn} = -c_{mn}^{ij,kl} = \delta_{jk}\delta_{im}\delta_{ln} - \delta_{li}\delta_{km}\delta_{jn}.$$

If we assume that the two algebras are paired by

$$\langle x^i, X_j \rangle = \delta_j^i, \qquad \langle y^{ij}, Y_{kl} \rangle = \delta_k^i \delta_l^j, \tag{4.16}$$

we can define a bilinear form on the vector space $s_+ \oplus s_-$ such that, in terms of (4.16), both $s_\pm$ are isotropic for it. Under these conditions we can consider the triple $(s_+, s_-, \bar{g} = s_+ + s_-)$. Indeed, by taking into account (2.6) we obtain the crossed commutation rules

$$[x^i, X_j] = 0, \qquad [x^i, Y_{jk}] = \tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})Y_{jk}, \qquad [y^{ij}, X_k] = \tfrac{1}{\sqrt{2}}(\delta_{ik} - \delta_{jk})y^{ij},$$
$$[y^{ij}, Y_{kl}] = \{\delta_{ik}(Y_{jl} + y^{lj}) - \delta_{jl}(Y_{ki} + y^{ik})\} - \delta_{ki}\delta_{lj}(X_i + x^i - X_j - x^j),$$

where $Y_{ij} \equiv 0$ and $y^{ij} \equiv 0$ for $i > j$.

We can avoid to check the compatibility conditions (2.5), since it is equivalent to the set of Jacobi identities for $\bar{g}$, which turns out to be a well-known Lie algebra and (2.5). This can be proven by considering the following change of basis:

$$H_i = \frac{1}{\sqrt{2}}(X_i + x^i), \qquad I_i = \frac{1}{\mathrm{i}\sqrt{2}}(X_i - x^i), \qquad F_{ij} = Y_{ij} + y^{ji},$$

and in this new basis the full set of commutation rules for $\bar{g}$ is

$$\begin{aligned} &[I_i, \cdot] = 0, \qquad [H_i, H_j] = 0, \qquad [H_i, F_{jk}] = (\delta_{ij} - \delta_{ik})F_{jk}, \\ &[F_{ij}, F_{kl}] = (\delta_{jk}F_{il} - \delta_{il}F_{kj}) + \delta_{jk}\delta_{il}(H_i - H_j). \end{aligned} \tag{4.17}$$

These relations are nothing but the commutation rules for $gl(n) \oplus t_n$, where $gl(n)$ is defined by the usual fundamental representation in terms of the $n \times n$ matrices H_i and F_{ij} $(i \neq j)$ defined as follows:

$$(H_i)^{jk} = \delta_i^j \delta_i^k, \qquad (F_{ij})^{kl} = \delta_i^k \delta_j^l, \qquad i, j, k, l = 1, \ldots, n.$$

Thus, $(s_+, s_-, \bar{g} = gl(n) \oplus t_n)$ is a self-dual Manin triple.

We stress that the choice of basis turns out to be essential in order to exhibit the Drinfeld double structure, since only those basis in $gl(n)$ that are consistent with the pairing (3.14) can be considered. This fact excludes the Chevalley basis (see [25]) for $gl(n)$.

4.1. Lie bialgebra structure

From (2.7), the co-commutator δ_D defining the canonical Lie bialgebra structure on $\bar{g}$ reads

$$\delta_D(I_i) = 0,$$

$$\delta_D(H_i) = 0,$$

$$\delta_D(F_{ij}) = -\frac{1}{2}F_{ij} \wedge (H_i - H_j) - \frac{\mathrm{i}}{2}F_{ij} \wedge (I_i - I_j) + \sum_{k=i+1}^{j-1} F_{ik} \wedge F_{kj}, \qquad i < j,$$

$$\delta_D(F_{ij}) = \frac{1}{2}F_{ij} \wedge (H_i - H_j) - \frac{\mathrm{i}}{2}F_{ij} \wedge (I_i - I_j) - \sum_{k=j+1}^{i-1} F_{ik} \wedge F_{kj}, \qquad i > j.$$

In particular, (s_+, δ_D) and its dual (s_-, δ_D) are Lie sub-bialgebras. The classical r-matrix (2.8), in the basis $\{H_i, F_{ij}, I_i\}$, is written as

$$\tilde{r} = \frac{1}{2}\sum_{i<j} F_{ji} \wedge F_{ij} + \frac{\mathrm{i}}{4}\sum_{i} H_i \wedge I_i = \tilde{r}_s + \tilde{r}_t.$$

Again $\tilde{r}_s$ generates the standard deformation of $gl(n)$ and $\tilde{r}_t$ is a twist (not of Reshetikhin type [31]). When all I_i are equal the twist $\tilde{r}_t$ becomes trivial.

Note also that the chain of Drinfeld doubles $\bar{g}_m \subset \bar{g}_{m+1}$ is preserved at the level of Lie bialgebras. However, although $gl(n)$ is a subalgebra of $\bar{g}$, the co-commutator $\delta_D(gl(n))$ does not define a Lie sub-bialgebra since it depends on the extra t_n sector.

Moreover, from the underlying Lie bialgebra $(\bar{g}, \delta_D)$ we see that, after quantization, the extra Abelian sector t_n gives rise to a set of twists that will intertwine with the standard quantization of the $gl(n)$ subalgebra (results concerning the known $gl(n)$ quantizations can be found in [32–36]). Moreover, from $U_z(gl(n) \oplus t_n)$ one would be able to recover $U_z(sl(n))$ in the representation of t_n in which all I_i are equal. Note also that while this 'central extension procedure' (1.1) can only be done on the complex, the final $U_z(sl(n))$ would be obtained on $\mathbb{R}$.

5. Concluding remarks

We have introduced a Drinfeld double structure on the Lie algebra $\bar{g} = gl(n) \oplus t_n$, inspired by the Cartan–Dynkin approach for the classification of simple Lie algebras in which the (solvable)

Borel subalgebras play an essential role. We briefly comment on some consequences of this construction both from a classical viewpoint and from the quantum deformation perspective.

Firstly, it is worth to emphasize that the choice of the basis in $\bar{g}$ is crucial because in order to have a Drinfeld double structure we need, up to a global factor, both an orthonormal basis in the Cartan subalgebra and a fixed normalization for all the remaining generators. For the $gl(n)$ generators, the pairing is given by the Killing form, i.e., for the Cartan subalgebra $K_{mn} = \delta_{m,n}$ while $K_{\alpha\beta} = \delta_{\alpha,-\beta}$ for the root vectors. This seems to be the natural choice and corresponds to the 'oscillator representation' convention [30]. For the t_n sector, the pairing is defined by imposing $\langle I_i, I_j \rangle = \delta_{ij}$.

Secondly, the results presented here can be thought of as a first step in order to approach quantum deformations of simple Lie algebras from a quantum double perspective. In particular, we have provided the first-order information for the construction of $U_z(gl(n) \oplus t_n)$ and, as a byproduct, of the Drinfeld–Jimbo deformation $U_z(sl(n))$. Under this approach the essential task is to obtain the full quantization of a solvable Lie algebra in which the only deformed coproducts correspond to the nilpotent generators. In this way, Serre relations do not play any role since all the root vectors are considered on the same footing and the quantization procedure is simplified by the self-dual nature of the underlying Lie bialgebra (and of the associated Poisson–Lie group). To achieve this quantization one could follow [37], where 6-dimensional Drinfeld doubles have been fully quantized through the method given in [38].

By following the same lines, the discussion of the Drinfeld double structure for the B, C and D series of simple Lie algebras will be presented elsewhere.

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References

[1] Drinfel'd V G 1983 *Sov. Math. Dokl.* **27** 68
[2] Drinfel'd V G 1987 Quantum groups *Proc. of the International Congress of Mathematicians (Berkeley, 1986)* ed A M Gleason (Providence, RI: American Mathematical Society)
[3] Klimcik C and Severa P 1995 *Phys. Lett.* B **351** 455
[4] Klimcik C and Severa P 1996 *Phys. Lett.* B **372** 65
[5] Lledó M A and Varadarajan V S 1998 *Lett. Math. Phys.* **45** 247
[6] von Unge R 2002 *J. High Energy Phys.* JHEP7(2002)14
[7] Hlavaty L and Snobl L 2002 *Mod. Phys. Lett.* A **17** 429
[8] Hlavaty L and Snobl L 2002 *Int. J. Mod. Phys.* A **17** 4043
[9] Snobl L 2002 *J. High Energy Phys.* JHEP9(2002)18
[10] Burroughs N 1990 *Commun. Math. Phys.* **127** 109
[11] Tjin T 1992 *Int. J. Mod. Phys.* A **7** 6175
[12] Majid S 1994 *J. Geom. Phys.* **13** 169
[13] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[14] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[15] Fronsdal C and Galindo A 1993 *Lett. Math. Phys.* **27** 59
[16] Ge M L and Liu X F 1992 *Lett. Math. Phys.* **24** 197
[17] Vladimirov A A 1993 *Mod. Phys. Lett.* A **8** 2573
[18] Sun C P 1993 *J. Math. Phys.* **34** 3440
[19] Sun C P, Li W and Ge M L 1993 *J. Phys. A: Math. Gen.* **26** 5449
[20] Sun C P and Ge M L 1993 *J. Phys. A: Math. Gen.* **26** 7031
[21] McAnally D S and Tsohantjis I 1997 *J. Phys. A: Math. Gen.* **30** 651

[22] Lyakhovsky V D and Tkach V I 1998 *J. Phys. A: Math. Gen.* **31** 2869
[23] Bernard D and Leclair A 1993 *Nucl. Phys.* B **399** 709
[24] Bais F A and Muller N M 1998 *Nucl. Phys.* B **530** 349
[25] Cornwell J F 1984 *Group Theory in Physics* (London: Academic)
[26] Drinfeld V G 1985 *Dokl. Akad. Nauk. SSSR* **283** 1060
[27] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[28] Belavin A A and Drinfeld V G 1983 *Funct. Anal. Appl.* **16** 159
[29] Gomez X 2000 *J. Math. Phys.* **41** 4939
[30] Frappat L, A Sciarrino and Sorba P 2000 *Dictionary on Lie Algebras and Superalgebras* (London: Academic)
[31] Reshetikhin N Yu 1990 *Lett. Math. Phys.* **20** 331
[32] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697
[33] Schirrmacher A 1991 *Z. Phys.* C **50** 321
[34] Alisauskas S and Smirnov Yu F 1994 *J. Phys. A: Math. Gen.* **27** 5925
[35] Dobrev V K and Parashar P 1993 *J. Phys. A: Math. Gen.* **26** 6991
Dobrev V K and Parashar P 1999 *J. Phys. A: Math. Gen.* **32** 443
[36] Fronsdal C and Galindo A 1995 *Lett. Math. Phys.* **34** 25
[37] Ballesteros A, Celeghini E and del Olmo M A 2005 *J. Phys. A: Math. Gen.* **38** 3909
[38] Ballesteros A, Celeghini E and del Olmo M A 2004 *J. Phys. A: Math. Gen.* **37** 1